

Charge Screening in the Finite Temperature Schwinger Model¹G. Grignani^a, G. Semenoff^b, P. Sodano^a and O. Tirkkonen^b

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Abstract

We compute the effective action and correlators of the Polyakov loop operator in the Schwinger model at finite temperature and discuss the realization of the discrete symmetries that occur there. We show that, due to nonlocal effects of massless fermions in two spacetime dimensions, the discrete symmetry which governs the screening of charges is spontaneously broken even in an effective one-dimensional model, when the volume is infinite. In this limit, the thermal state of the Schwinger model screens an arbitrary external charge; consequently the model is in the deconfined phase, with the charge of the deconfined fermions completely screened. In a finite volume we show that the Schwinger model is always confining.

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1 Introduction

The Schwinger model [1] is an exactly solvable model which describes the electrodynamics of a massless fermion in 1+1 spacetime dimensions. It is the classic example of a confining gauge theory and displays some of the features of quantum chromodynamics or other higher dimensional gauge theories where strong infrared effects are important [2]. In one space dimension the tree level electron-positron potential, $V(x) \propto e^2|x|$, is already confining without quantum fluctuations. Detailed analysis in both the path integral [3, 4] and operator methods [5, 6, 7] shows that the spectrum is completely gapped, exhibits chiral symmetry breaking and has no charged excitations.

Over the past few decades, there has been considerable interest in the properties of the Schwinger model at finite temperature, both in the massless [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and in the massive [21] case. The most interesting question as to whether the chiral symmetry breaking seen in the vacuum is restored at high temperatures was answered in the negative long ago [22]. The breaking of the chiral symmetry is a consequence of the axial anomaly, rather than spontaneous symmetry breaking, and the axial anomaly, being a result of short distance physics, is insensitive to temperature. Thus, many of the features of the Schwinger model are not changed by temperature.

In this paper, we advocate the use of temperature to explore the spectrum of the Schwinger model. The thermal state is a density matrix with non-vanishing contributions from all states in the spectrum with finite energy and thus contains information about all of the states.

As an infrared regularization, we shall consider the space as a circle of circumference L on which all of the basic fields of the Schwinger model have periodic boundary conditions. The Hamiltonian quantization in this regularization and some of the questions concerning topology and theta-vacua which arise in this case were discussed by Manton [23]. These issues, as well as questions of Wilson line phase dynamics and correlators, were further developed in Ref. [24].

Part of our motivation for this work is to test a recent idea [25] that the Polyakov loop operator, introduced by Polyakov [26] and Susskind [27] as an order parameter for confinement in non-abelian Yang-Mills theory in higher dimensions is also a useful operator for Abelian gauge theory. In non-Abelian gauge theory, the Polyakov loop has the limitation that it is an order parameter for confinement only in models where all of the fields are invariant under global gauge transforms in the center of the gauge group, i.e. are in the adjoint or some other zero ‘N-ality’ representation. In electrodynamics, on the other hand, it can be used in any model which is essentially compact in the sense that all of the charges of the dynamical fields are integer multiples of some basic charge [25]. Then, as was argued in [25], the Polyakov loop with an incommensurate charge can be used to probe the ability of the electrodynamic system to screen external charges.

1.1 Z_N symmetry of finite temperature Yang-Mills theory

We shall first review the role of the Polyakov loop operator as an order parameter for confinement in Yang-Mills theory at finite temperature. This is conventionally seen in the Euclidean path integral formulation of the finite temperature gauge theory. In that formulation, the Polyakov loop operator measures the holonomy of the gauge connection in the periodic Euclidean time,

$$P(\vec{x}) \equiv \text{tr} \mathcal{P} \exp \left(i \int_0^{1/T} d\tau A_0(\tau, \vec{x}) \right) \quad (1)$$

whose correlators in a finite temperature Yang-Mills theory are defined by the Euclidean path integral

$$\begin{aligned} & \langle P(\vec{x}_1) \dots P(\vec{x}_m) P^\dagger(\vec{y}_1) \dots P^\dagger(\vec{y}_n) \rangle \\ &= \frac{\int dA_\mu e^{-\int \text{tr} F^2/4} P(\vec{x}_1) \dots P(\vec{x}_m) P^\dagger(\vec{y}_1) \dots P^\dagger(\vec{y}_n)}{\int dA_\mu e^{-\int_0^{1/T} \text{tr} F^2/4}} \end{aligned} \quad (2)$$

where the gauge field has periodic boundary conditions,

$$A_\mu(1/T, \vec{x}) = A_\mu(0, \vec{x}) \quad (3)$$

Since the Yang-Mills field transforms in the adjoint representation of the gauge group,

$$A'_\mu(\tau, \vec{x}) = g^{-1}(\tau, \vec{x}) A_\mu(\tau, \vec{x}) g(\tau, \vec{x}) + i g^{-1}(\tau, \vec{x}) \nabla_\mu(\tau, \vec{x}) g(\tau, \vec{x}) \quad (4)$$

they remain periodic under gauge transformations which are periodic up to an element, Z , of the center of the group,

$$g(1/T, \vec{x}) = Z g(0, \vec{x}) \quad (5)$$

The center of $SU(N)$ is Z_N , the additive group of the integers modulo N , whereas the center of $U(N)$ is the Abelian group $U(1)$. The coset of the group of all gauge transformations modulo those which are strictly periodic is a global transformation by elements in the center of the gauge group.

As well as pure Yang-Mills theory, any gauge theory which has matter fields which transform in the adjoint, or any other zero ‘N-ality’ representation of the gauge group will have this symmetry of the path integral. Furthermore, this symmetry exists for any gauge group which has a non-trivial center. In the following, we shall take the gauge group to be $SU(N)$, with the center Z_N . The non-trivial topological structure arises from the non-vanishing first homotopy $\Pi_1(SU(N)/Z_N) = Z_N$. An important question is whether or not this Z_N symmetry is spontaneously broken at finite temperature.

Under the gauge transformation (5), the Polyakov loop operator transforms as

$$P'(\vec{x}) = Z P(\vec{x}) \quad (6)$$

Therefore, this operator can be used as an order parameter for breaking of the Z_N symmetry.

The connection between the breaking of Z_N symmetry and confinement is through the fact that the correlators of Polyakov loop operators

$$e^{-F(\vec{x}_1, \dots, \vec{x}_m, \vec{y}_1, \dots, \vec{y}_n)/T} = \langle P(\vec{x}_1) \dots P(\vec{x}_m) P^\dagger(\vec{y}_1) \dots P^\dagger(\vec{y}_n) \rangle \quad (7)$$

can be interpreted as giving the free energy $F(\vec{x}_1, \dots, \vec{x}_m, \vec{y}_1, \dots, \vec{y}_n)$ of the finite temperature gauge theory with an array of classical, external, fundamental representation quark sources at positions $\vec{x}_1, \dots, \vec{x}_m$ and anti-quark sources at positions $\vec{y}_1, \dots, \vec{y}_n$. The normalization of the correlator subtracts the free energy of the gauge theory at the same temperature in the absence of sources.

If the Z_N symmetry is not spontaneously broken, the correlator in (7) vanishes unless $m = n$ modulo N , i.e. unless the quarks and anti-quarks occur in the right numbers to make up mesons, which are quark-antiquark pairs, or baryons or anti-baryons, which are groups of N quarks or N anti-quarks, respectively. The vanishing of the correlator is interpreted as the quark charge distribution having infinite free energy when it has quantum numbers which cannot be combined into color singlets, i.e. as quark confinement. On the other hand, if the Z_N symmetry is spontaneously broken, the correlators can be non-zero and even quark charge distributions which cannot form color singlets can have finite free energy.

Furthermore, the free energy $F(\vec{x}_1, \dots, \vec{x}_m, \vec{y}_1, \dots, \vec{y}_n)$ can be viewed as the effective potential energy of the array of quarks and antiquarks. For example, the effective interaction between a quark and anti-quark is given by $F(\vec{x}, \vec{y})$. If this correlator increases with distance, as it would in a confined phase where there is a non-zero string tension, then the correlator of Polyakov loop operators would have the clustering property

$$\lim_{|x-y| \rightarrow \infty} e^{-F(\vec{x}, \vec{y})/T} = \lim_{|x-y| \rightarrow \infty} \langle P(\vec{x}) P^\dagger(\vec{y}) \rangle = 0 \quad (8)$$

This implies that the Z_N symmetry is unbroken and is consistent with the vanishing of the expectation value of the loop operator,

$$\langle P(\vec{x}) \rangle = 0 \quad (9)$$

On the other hand, in the deconfined phase, one would expect the quark-antiquark potential to fall off to zero with some screening length (the non-Abelian analog of Debye screening). In that case,

$$\lim_{|x-y| \rightarrow \infty} e^{-F(\vec{x}, \vec{y})/T} = \lim_{|x-y| \rightarrow \infty} \langle P(\vec{x}) P^\dagger(\vec{y}) \rangle \neq 0 \quad (10)$$

This implies that the Z_N symmetry is spontaneously broken and is consistent with the loop operator having a non-vanishing expectation value

$$\langle P(\vec{x}) \rangle \neq 0 \quad (11)$$

To properly compute this one-point function, one should as usual introduce a symmetry breaking external field through a term in the action such as

$$S_\lambda = \lambda \int d\vec{x} \left(P(\vec{x}) + P^\dagger(\vec{x}) \right) \quad (12)$$

and compute the one-point function, which would be non-zero when λ is not zero. Then, the occurrence of symmetry breaking would be seen when the limit

$$\lim_{\lambda \rightarrow 0} \langle P(\vec{x}) \rangle_\lambda \quad (13)$$

is non-zero.

This formalism is well developed for finite temperature Yang-Mills theory and some other pure gauge theories such as compact $U(1)$ and some Z_N gauge theory. All non-trivial pure gauge theories in spacetime dimensions greater than two exhibit a high-temperature de-confined phase and almost all have a phase transition to a confined phase at some critical temperature. Details are summarized in the comprehensive review by Svetitsky, [28].

However, for gauge theories with dynamical quarks, the Polyakov loop operator is not a useful order parameter to characterize a confining phase. The reason is that, since the quark fields transform in the fundamental representation of the gauge group, their action and boundary conditions in the path integral are invariant under only strictly periodic gauge transformations. Thus, fundamental representation quarks are said to break the Z_N symmetry explicitly. The free energy of a distribution of external quarks is always finite. This is interpreted as the possibility of pair creation of dynamical quark-antiquark pairs so that the dynamical quarks can screen the color of any external distribution of quark or anti-quark sources.

1.2 Z symmetry of quantum electrodynamics

In an Abelian gauge field theory, the Polyakov loop operator is defined by the analog of (1)

$$P(\vec{x}) = \exp \left(i \int_0^{1/T} d\tau A_0(\tau, x) \right) \quad (14)$$

and its correlators are computed using the finite temperature path integral of quantum electrodynamics

$$\langle \prod_i P_{e_i}(\vec{x}_i) \rangle = \frac{\int dA_\mu d\psi d\bar{\psi} e^{-\int_0^{1/T} F_{\mu\nu}^2/4 + \bar{\psi}(\gamma \cdot D + m)\psi} \prod_i e^{ie_i \int_0^{1/T} d\tau A_0(\tau, \vec{x}_i)}}{\int dA_\mu d\psi d\bar{\psi} e^{-\int_0^{1/T} F_{\mu\nu}^2/4 + \bar{\psi}(\gamma \cdot D + m)\psi}} \quad (15)$$

with the (anti-)periodic boundary conditions,

$$A_\mu(1/T, \vec{x}) = A_\mu(0, \vec{x}) \quad (16)$$

$$\psi(1/T, \vec{x}) = -\psi(0, \vec{x}) \quad , \quad \bar{\psi}(1/T, \vec{x}) = -\bar{\psi}(0, \vec{x}) \quad (17)$$

These boundary conditions, as well as the measure and action in the functional integral are invariant under the gauge transformation

$$A'_\mu(\tau, \vec{x}) = A_\mu(\tau, \vec{x}) + \nabla_\mu \chi(\tau, \vec{x}) \quad (18)$$

$$\psi'(\tau, \vec{x}) = e^{ie\chi(\tau, \vec{x})}\psi(\tau, \vec{x}) \quad , \quad \bar{\psi}'(\tau, \vec{x}) = e^{-ie\chi(\tau, \vec{x})}\bar{\psi}(\tau, \vec{x}) \quad (19)$$

when the gauge function is periodic up to an additive constant

$$\chi(1/T, \vec{x}) = \chi(0, \vec{x}) + 2\pi n/e \quad (20)$$

where n is an integer. The coset of the group of all allowed gauge transformations modulo the group of all strictly periodic gauge transformations is $\Pi_1(U(1)) = Z$, the additive group of the integers. Note that this is a symmetry of the functional integral representation of the partition function even in the presence of dynamical electrons. It is an interesting and well-defined question to ask whether this symmetry is realized in a spontaneously broken or unbroken phase in quantum electrodynamics.

The Abelian Polyakov loop operator transforms under a gauge transformation as

$$P'_\tilde{e}(\vec{x}) = e^{2\pi n \tilde{e}/e} P_\tilde{e}(\vec{x}) \quad (21)$$

and is not invariant under the Z symmetry unless \tilde{e} is an integer multiple of the electron charge e . This transformation law was noted by Hansson, Nielsen and Zahed [19] when the incommensurate charge \tilde{e} was a fraction of the electron charge and the Polyakov loop operator transforms under a Z_N subgroup of Z . Thus, as in finite temperature Yang-Mills theory, the Abelian version of the Polyakov loop operator can be used as an order parameter for the Z symmetry.

The Z symmetry is related to charge screening and confinement in quantum electrodynamics in the same way as the Z_N symmetry of finite temperature Yang-Mills theory. The correlators of Polyakov loop operators measure the free energy of the electrodynamic system in the presence of a distribution of static charged sources. The two-point function, for example,

$$e^{-F_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y})/T} = \langle P_\tilde{e}(\vec{x}) P_{-\tilde{e}}(\vec{y}) \rangle \quad (22)$$

measures the effective interaction potential between particles with charges \tilde{e} and $-\tilde{e}$ and positions \vec{x} and \vec{y} , respectively. In a de-confined phase, we would expect Debye screening and the asymptotic form of the potential at large separations to decay exponentially with the Debye mass of the photon. This would imply that the correlator of two Polyakov loop operators approaches a constant at large separations. This implies spontaneous breaking of the Z symmetry.

In a confined phase, there should be a string tension, and $F_{\tilde{e}, -\tilde{e}}(\vec{x}, \vec{y})$ increases with separation. This would give a decay of the two-point correlator of Polyakov loops consistent with a Z symmetric phase.

In 3+1-dimensional quantum electrodynamics, we would expect that, at least in the physically observed weak coupling regime, the Z symmetry is broken spontaneously at all temperatures. It has recently been argued [25] that in 2+1-dimensional parity invariant electrodynamics, at least if the electron mass is large enough, both the confined and de-confined phases should exist with a Kosterlitz-Thouless type of phase transition between them at some finite temperature. In the following we shall examine the case of 1+1-dimensional electrodynamics. There, when the mass of the electron is non-zero, the dimension of the space is too low to allow a phase transition. Based on the results of [29] one can expect that the theory exists in a confined phase at any temperature (except for some specific θ -vacua found in [30]).

On the other hand, we will be able to show that the Z symmetry is spontaneously broken in 1+1-dimensional electrodynamics when the electron is massless, i.e. in the Schwinger model.

1.3 Z symmetry of the Schwinger model

In this paper we shall examine the expectation value of the Polyakov loop operator in the Schwinger model. The Z symmetry transforms the temporal component of the gauge field as

$$A_0 \rightarrow A_0 + 2\pi n T/e, \quad n \in Z. \quad (23)$$

We shall give an interpretation for the Z symmetry in the Hamiltonian formalism in terms of the quantization of charge, in states of the thermal ensemble. If Z is a good symmetry, all charged states through which the thermal system fluctuates have charges which are quantized in units of the electron charge. If Z is spontaneously broken, there are quantum states available which have arbitrary charge. If Z were broken to a subgroup, Z_N , this would imply that there were fractionally charged states with charges quantized in units of e/N where e is the charge of the dynamical electron. An explicit realization of the latter breaking pattern may be of relevance for applications to one dimensional condensed matter systems.

Our analysis of the finite temperature Schwinger model with one flavor of fermions leads us to the following results:

i.) In the one-dimensional Coulomb gas, which can be regarded as a certain limit of electrodynamics which has very massive charged particles, the Z symmetry breaking problem resembles that of the quantum pendulum problem, or 1-dimensional sine-gordon theory. The Z symmetry is unbroken at all temperatures, corresponding to a confining state.

ii.) In the Schwinger model where the space is a circle with circumference L and with periodic boundary conditions for both the photon and electron fields, we compute the expectation value of the Polyakov loop operator and its correlators. We find that the expectation value of the Polyakov Loop operator with electric charge \tilde{e} an integer

multiple of the electron charge is a non-zero computable constant. When the charge in the loop operator is not an integer multiple of the electron charge, the expectation value of the Polyakov loop operator vanishes,

$$\left\langle \exp \left\{ i\tilde{e} \int_0^{1/T} d\tau A_o(\tau, x) \right\} \right\rangle = 0 \quad \text{if } \tilde{e} \neq \text{integer} \cdot e \quad (24)$$

at all temperatures T . This can be seen as the consequence of the discrete symmetry (23) which is realized in an unbroken phase when the volume is finite.

iii.) In the infinite volume limit, the Z symmetry is spontaneously broken. This is seen by examining the following limits:

$$\lim_{|x-y| \rightarrow \infty} \left(\lim_{L \rightarrow \infty} \left\langle e^{i\tilde{e} \int_0^{1/T} d\tau A_o(\tau, x)} e^{-i\tilde{e} \int_0^{1/T} d\tau A_o(\tau, y)} \right\rangle \right) = \text{constant} \neq 0 \quad (25)$$

for any charge \tilde{e} and at all temperatures. We interpret this as implying that the thermal state of the Schwinger model can screen arbitrary external charges. The exact form of the correlator is known and the asymptotic, exponential decay of the correlation function is governed by the Schwinger mass of the photon, $m_s^2 = e^2/\pi$.

1.4 Symmetry breaking in one dimension?

The result that the Z symmetry is spontaneously broken was anticipated by Hansson, Nielsen and Zahed [19]. It is surprising in the sense that, as we shall argue in the following sections, the effective action for the Polyakov loop operator is an one dimensional field theory with a discrete Z symmetry. Normally such symmetries cannot be spontaneously broken, as the long-range correlations described by (25) are forbidden by the accompanying strong infrared effects. From another viewpoint, the ordered state of the broken symmetry theory is unstable to the condensation of domain walls.

This can be understood by a simple argument: If we consider a one dimensional system with N sites and n domain walls, the entropy of the state can be estimated by noting that the domain walls can be arranged in $\binom{N}{n}$ ways, leading to entropy

$$S = \ln \left(\binom{N}{n} \right) \quad (26)$$

If the domain wall has energy ϵ the free energy at temperature \bar{T} for large N and n is then given by

$$F = n\epsilon + \bar{T} \left(n \ln n + (N - n) \ln(N - n) - N \ln N \right) \quad (27)$$

Note that for all values of the domain wall energy ϵ and temperature \bar{T} , the entropy always grows faster than the energy as n is increased. This leads to a condensation of domain walls. The equilibrium number of domain walls has a Fermi-Dirac distribution

$$\langle n \rangle = N \frac{e^{-\epsilon/\bar{T}}}{1 + e^{-\epsilon/\bar{T}}} \quad (28)$$

If the size of the system is Na where a is the lattice spacing, the correlation length is of order the mean distance between domain walls,

$$\xi \approx (1 + e^{\epsilon/\bar{T}}) a \quad (29)$$

which is always small, of order the “lattice spacing” or inverse ultraviolet cutoff. Thus, domain wall condensation would seem to always destroy one-dimensional order.

We shall argue that the Schwinger model evades domain wall condensation at all finite temperatures by having domain walls with infinite energy. This occurs because the domain walls are actually instantons in a static gauge. The fermion determinant vanishes on instanton configurations, giving the instantons an infinite free energy. Thus, the only way out of the above argument, that $\epsilon/\bar{T} = \infty$, is actually realized in the Schwinger model.

When the electrons have a mass, one can expect that the domain wall energy for small mass diverges logarithmically, $\epsilon \approx -\bar{T} \ln(m/\mu)$, for small m where μ is a dimensional constant related to the fermion mass m and the confining scale which is given by the electric charge e . Thus, if the electron in the Schwinger model had non-zero mass the domain walls would have finite energy, the correlation length would be

$$\xi \approx (1 + \mu/m) a \quad (30)$$

and the domain wall condensation would ruin the symmetry breaking at all temperatures, apparently even in the zero temperature limit.

1.5 Deconfinement versus superconductivity

There is another interpretation of the physical state of the Schwinger model alternative to confinement. The fact that the photon has a mass can be interpreted as the Schwinger model being a superconductor or, since in one dimension there is no possibility of magnetic fields and therefore no Meissner effect, a perfect conductor. This is seen by considering the current induced in the Schwinger model ground state by an external electric field which can be obtained from the exact identities for current conservation

$$\nabla_\mu \langle j_\mu(x) \rangle_A = 0 \quad (31)$$

and the axial anomaly equation which can be presented as

$$\epsilon_{\mu\nu} \nabla_\mu \langle j_\nu \rangle_A = \frac{e^2}{2\pi} \epsilon_{\mu\nu} \nabla_\mu A_\nu^{\text{ext}} \quad (32)$$

which makes use of the kinematical identity relating the axial and vector currents of two dimensional fermions

$$\langle j_\mu^5 \rangle_A = i \epsilon_{\mu\nu} \langle j_\nu \rangle_A \quad (33)$$

The above equations have the solution

$$\langle j_\mu(x) \rangle_A = \frac{e^2}{\pi} \frac{\epsilon_{\mu\nu} \nabla_\nu}{-\nabla^2} E^{\text{ext}} \quad (34)$$

in terms of the external electric field, E^{ext} . This is a superconducting response.

For example, if the electric field is spatially constant, it has the solution

$$\langle J_0 \rangle_A = 0 \quad , \quad \langle J_1 \rangle_A = \frac{e^2}{\pi} E^{\text{ext}} t \quad (35)$$

where the current increases linearly with time.

This superconducting response can lead to a super-screening of electric fields which would otherwise be caused by external charges. We propose this as an alternative to the other obvious interpretation of the breaking of the Z symmetry, the loss of confinement.

Our results regarding the Z symmetry breaking support the conclusions of Refs. [31, 30, 32], where the concept of screening versus confinement in 1+1 dimensional field theories was discussed. There, deconfinement was interpreted as arising from liberation of so called bleached states, with the charges of the deconfined fermions being completely screened. The “bleached states” were originally found in the operatorial solution of the Schwinger model in [5]. In later works [6, 33] they were argued to be unphysical in that they are created by non-local operators and therefore are unattainable by the action of operators within the algebra of local observables. The strong infrared effects driving the theory in infinite volume are essentially non-local, however, so we shall interpret our results as indicating the possible emergence of “bleached states” in the finite temperature infinite volume limit.

In the next Section, we shall present two simple examples where the realization of the Z symmetry is in the unbroken phase. In the subsequent Section we shall review the Hamiltonian formulation of the Schwinger model. It is somewhat independent of the rest of the paper and is intended mainly to fix notation and remind the reader of the standard picture. In Section 4, we describe the path integral representation of the partition function of the Schwinger model at finite temperature. We also introduce the effective action for the Polyakov loop operator and make explicit the physical interpretation of the Z symmetry. In Section 5, we calculate the Polyakov loop expectation values and prove results ii) and iii). In Section 6, we present a discussion of our conclusions.

2 Two Simple Examples

Before we solve for the Polyakov loop correlator in the Schwinger model, let us consider the following examples.

2.1 Free electrodynamics in 1+1 dimensions

First, let us consider the case of two dimensional pure U(1) gauge theory. The correlator of Polyakov loop operators is given by

$$\langle \prod_j e^{ie_j \int_0^{1/T} d\tau A_0(\tau, x_j)} \rangle \equiv \frac{\int dA_\mu(x) e^{-\int_0^{1/T} dx d\tau F_{01}^2/2} \prod_j e^{ie_j \int_0^{1/T} d\tau A_0(\tau, x_j)}}{\int DA_\mu(x) e^{-\int_0^{1/T} dx d\tau F_{01}^2/2}} \quad (36)$$

where the finite temperature path integral is done with periodic boundary conditions. The partition function has the formal symmetry

$$A_0(x, \tau) \rightarrow A_0(x, \tau) + \text{constant} \quad (37)$$

which, because of the absence of the dynamical electron field, is larger than the Z symmetry.

It is straightforward to perform the gaussian integral in (36) to obtain the exponential of the 1-dimensional Coulomb energy. The result has an infrared singularity unless

$$\sum e_j = 0 \quad (38)$$

If this constraint is obeyed, we obtain the expression

$$\langle \prod_j e^{ie_j \int d\tau A_0(x_j, \tau)} \rangle = \exp \left(- \sum_{ij} \frac{e_i e_j}{2T} |x_i - x_j| \right) \quad (39)$$

This is the usual confining 1-dimensional coulomb potential. It corresponds to a state where the symmetry under translation of A_0 in the path integral is unbroken. This is seen, for example, in the correlator $\langle e^{ie \int A_0(\tau, x)} e^{-ie \int A_0(\tau, y)} \rangle$ which has the asymptotic form

$$\lim_{|x-y| \rightarrow \infty} \langle e^{ie \int A_0(\tau, x)} e^{-ie \int A_0(\tau, y)} \rangle = \lim_{|x-y| \rightarrow \infty} e^{-e^2 |x-y|} = 0 \quad (40)$$

The cluster decomposition implies that the symmetry is unbroken at any finite temperature.

2.2 1-Dimensional Coulomb gas

The 1-dimensional Coulomb gas is implicitly solvable through the Gibbs ensemble calculation of Ref. [34]. Here, we obtain the grand canonical partition function of a neutral Coulomb gas by the following construction. Consider the statistical mechanics of a state with m classical particles with charge e occupying positions x_1, \dots, x_m and n classical particles with charge $-e$ occupying positions y_1, \dots, y_n . The partition function is given by

$$\frac{\int dA_\mu e^{-\int_0^{1/T} F_{\mu\nu}^2/4} e^{ie \int_0^{1/T} d\tau (\sum_1^m A_0(x_j) - \sum_1^n A_0(y_j))}}{\int dA_\mu e^{-\int_0^{1/T} F_{\mu\nu}^2/4}} \quad (41)$$

We multiply by the statistical factor for identical particles, $1/m!n!$ and a fugacity paramter λ^{m+n} , average over positions x_i and y_i by integrating and sum over m and n to obtain the partition function

$$Z[\lambda, T] = \frac{\int dA_\mu e^{-\int_0^{1/T} F_{\mu\nu}^2/4 + \lambda \int dx e^{ie \int_0^{1/T} d\tau A_0(\tau, x) + \text{c.c.}}}}{\int dA_\mu e^{-\int_0^{1/T} F_{\mu\nu}^2/4}} \quad (42)$$

If we fix the gauge

$$\frac{\partial}{\partial \tau} A_0(\tau, x) = 0 \quad (43)$$

we can do the integral over A_1 and obtain the one-dimensional sine-gordon theory

$$Z[\lambda, T] = \int \prod_x dA_0(x) \exp \left(- \int dx \left(\frac{T}{2} (dA_0(x)/dx)^2 - \lambda \cos(eA_0(x)/T) \right) \right) \quad (44)$$

The effective action for $A_0(x)$ explicitly has the symmetry under the shift $A_0 \rightarrow A_0 + 2\pi n T/e$. In the one-dimensional system (44) this symmetry cannot be spontaneously broken for any values (aside from zero or infinity) of the parameters λ and T . Thus, the expectation value of the Polyakov loop must vanish unless it has charge e . This we interpret as confinement. There is no confinement-deconfinement transition in this model.

3 Hamiltonian Formalism

3.1 Hamiltonian, gauge constraints and theta-states

We shall consider 1+1-dimensional electrodynamics defined on a compact space, $x \in [0, L]$. We begin by working in the canonical, Hamiltonian formalism. The Hamiltonian is derived from the action

$$S = \int dt \int_0^L dx \left(\frac{1}{2} F_{01}^2 - \bar{\psi} \gamma \cdot (i\nabla + eA) \psi \right) \quad (45)$$

In this action, the canonical momentum conjugate to the spatial component of the gauge field ($A_1(t, x)$, which we shall shortly rename $A(t, x)$) is the electric field $E(t, x) \equiv F_{01}(t, x) = \nabla_t A_1 - \nabla_x A_0$. The momentum conjugate to the fermion $\psi(t, x)$ is $i\psi^\dagger(t, x)$. The non-vanishing equal time (anti-) commutation relations are therefore

$$\begin{aligned} [A(x), E(y)] &= i\delta(x - y) \\ \{\psi(x), \psi^\dagger(y)\} &= \delta(x - y) \end{aligned} \quad (46)$$

The temporal component of the gauge field, $A_0(t, x)$, appears in the action (45) without time derivatives and acts as a Lagrange multiplier to impose the constraint of invariance under gauge transformations. The Hamiltonian is obtained as

$$H = \int_0^L dx \left(\frac{1}{2} E^2(x) + \psi^\dagger(x) \alpha (i\nabla + eA(x)) \psi(x) \right) \quad (47)$$

where $\alpha = \gamma_5 = \gamma_0\gamma_1$ is a 2×2 Hermitean Dirac matrix and $\nabla \equiv d/dx$. The massless Dirac Hamiltonian can be decomposed into Hamiltonians for left and right movers as

$$H_{\text{Dirac}} = \int_0^L dx \left(\psi_L^\dagger(x)(i\nabla + eA(x))\psi_L(x) - \psi_R^\dagger(x)(i\nabla + eA(x))\psi_R(x) \right) \quad (48)$$

All fields have periodic boundary conditions in space,

$$\begin{aligned} A(L) &= A(0) & E(L) &= E(0) \\ \psi(L) &= \psi(0) & \psi^\dagger(L) &= \psi^\dagger(0) \end{aligned} \quad (49)$$

The Hamiltonian and commutation relations must be supplemented by the first class constraint, or Gauss' law, which is the operator obtained by taking a functional derivative of the action (45) by A_o ,

$$G(x) \equiv -\nabla E(x) - e\psi^\dagger(x)\psi(x) \sim 0 \quad (50)$$

and ensuring that the quantum states are invariant under time-independent gauge transformations. The latter are generated by the operator

$$G[\chi] \equiv \int_0^L dx \chi(x) G(x) \quad (51)$$

where $\chi(x)$ is a periodic function, $\chi(L) = \chi(0)$. The action of the operator (51) is

$$\begin{aligned} e^{iG[\chi]} A(x) e^{-iG[\chi]} &= A(x) + \nabla \chi(x) \\ e^{iG[\chi]} E(x) e^{-iG[\chi]} &= E(x) \\ e^{iG[\chi]} \psi(x) e^{-iG[\chi]} &= e^{ie\chi(x)} \psi(x) \\ e^{iG[\chi]} \psi^\dagger(x) e^{-iG[\chi]} &= e^{-ie\chi(x)} \psi^\dagger(x). \end{aligned} \quad (52)$$

This is a symmetry of the Hamiltonian (47) and of the commutation relations (46), which preserves the boundary conditions (49).

There is a larger class of gauge transformations under which the Hamiltonian and commutation relations are invariant, which preserve the boundary conditions (49) and which are not generated by the Gauss operator $G[\chi]$. These have gauge functions which are not strictly periodic, but obey the condition

$$\chi_n(x+L) = \chi_n(x) + 2\pi n/e. \quad (53)$$

This guarantees that both the electron operator and the gauge field boundary condition is unchanged. Such 'large' gauge transformations can always be expressed as a periodic gauge transformation plus a representative of the large, non-periodic transformations as

$$\chi_n(x) = \chi_o(x) + 2\pi nx/Le, \quad (54)$$

where $\chi_o(x)$ is periodic.

Large gauge transformations are implemented by an unitary operator $\exp(iG_\ell[\chi_n])$, where the Hermitean operator is

$$G_\ell[\chi_n] = \int_0^L \left(\nabla \chi_n(x) E(x) - e \chi_n(x) \psi^\dagger(x) \psi(x) \right) \quad (55)$$

Using (54), this generator can be written as a large gauge transformation generator and a Gauss' operator

$$G_\ell[\chi_n] = G[\chi_o] + \frac{2\pi n}{Le} \int_0^L dx E(x) - \frac{2\pi n}{L} \int_0^L dx x \psi^\dagger(x) \psi(x) \quad (56)$$

The quantization of the model with commutator algebra (46), Hamiltonian (47) and constraint (50) can proceed in two different ways. First, one can solve the constraint (50) at the classical level by imposing an auxiliary condition on the remaining degrees of freedom. The second and equivalent approach, which we shall pursue in the following, is to quantize the dynamical system specified by (47) and (46) as it is. Then, on the Hilbert space which represents the algebra (46) and where the Hamiltonian is diagonalizable, we shall impose the physical state condition

$$G[\chi_o] |\text{physical state}\rangle = 0 \quad (57)$$

The physical states are thus invariant under all periodic gauge transformations.

However, they need not be invariant under the set of all gauge transformations. In fact, it is only necessary that they transform under a unitary irreducible representation of the coset of time-independent gauge transformations modulo the periodic ones. The coset group is isomorphic to the translation group of the integers, Z , whose unitary irreducible representations are one dimensional phases, $e^{i\theta n}$. Thus, if we implement a large gauge transformation using the operator $G_\ell[\chi_n]$, the physical states should transform as

$$e^{iG_\ell[\chi_n]} |\text{physical state}, \theta\rangle = e^{i\theta n} |\text{physical state}, \theta\rangle \quad (58)$$

In this way, the physical states are characterized by a theta-angle.

Like the theta-angle of non-Abelian gauge theories in four spacetime dimensions [35, 36], there exists a canonical transformation which removes the theta-angle from the states and introduces a theta term in the action. The unitary operator which implements this transformation is

$$U(\theta) = \exp \left(-i\theta e/2\pi \int_0^L dx A(x) \right) \quad (59)$$

In the new system, the theta angle is absent from the physical states and the electric field operator is altered. The new hamiltonian is

$$H = \int_0^L dx \left(\frac{1}{2} (E(x) + \theta e/2\pi)^2 + \psi^\dagger(x) \alpha (i\nabla + eA) \psi(x) \right) \quad (60)$$

In this way, one sees that θ has the interpretation of a constant background electric field, as discussed by Coleman, Jackiw and Susskind [29]. In the original action, taking into account the role of A_0 , (45) is modified by the addition of the conventional θ term,

$$S = \int d^2x \left(\frac{1}{2} F_{01}^2 + \theta F_{01} - \bar{\psi} \gamma \cdot (i\nabla + eA) \psi(x) \right) \quad (61)$$

It turns out that, in the massless Schwinger model, the physical states do not depend on θ . It is also possible to see that the parameter θ , which appears with the topological term, in the action is invariant. In the following we shall retain the θ -dependence in order to demonstrate the θ -independence of the partition function.

In the next Section we shall discuss the construction of the path integral representation of the thermodynamic partition function.

4 Path integral representation of the partition function

The thermodynamic description of field theory is most conveniently obtained from the partition function which for a gauge theory is gotten by taking a trace over physical states of the Gibbs distribution, $e^{-H/T}$, where T is the temperature and we work in units where the Boltzmann's constant as well as the Planck's constant and the speed of light are equal to one. In constructing the partition function it is convenient to consider all the states which represent the commutator algebra (46) and insert a projection operator which projects over the physical states, and onto a sector with a fixed vacuum angle θ . The trace is thus given using a complete set of states which represent (46),

$$Z[T] = \sum_s \langle s | e^{-H/T} P_\theta | s \rangle \quad (62)$$

The appropriate projection operator is constructed from the unitary operator which generates gauge transformations

$$P_\theta = \frac{1}{\text{Vol.G}} \sum_n e^{-in\theta} \int [d\chi_n] e^{iG[\chi_n]} \quad (63)$$

where we have integrated over all time independent gauge transformations and divided by the (infinite) volume of the gauge group. This results in the expression for the partition function

$$Z[T] = \frac{1}{\text{Vol.G}} \sum_n e^{-in\theta} \int [d\chi_n] e^{-S_{\text{eff}}[\chi_n]} \quad (64)$$

where the effective action for the gauge degrees of freedom is given by the twisted trace

$$e^{-S_{\text{eff}}[\chi]} = \sum_s \langle s | e^{-H/T} | s^{\chi_n} \rangle \quad (65)$$

This effective action has a standard path integral representation; in phase space it is given by

$$e^{-S_{\text{eff}}[\chi]} = \int \prod_{x \in [0, L]} \prod_{\tau \in [0, 1/T]} d\psi(\tau, x) d\psi^\dagger(\tau, x) dA(\tau, x) dE(\tau, x) e^{-S_E[\psi, \psi^\dagger, A, E]} \quad (66)$$

where the Euclidean first order action is

$$S_E = \int_0^{1/T} d\tau \int_0^L dx \left(iE\dot{A} + \frac{1}{2}E^2 - \psi^\dagger [\nabla_\tau + i\alpha\nabla_x + e\alpha A] \psi \right). \quad (67)$$

The electric field $E(\tau, x)$ has open boundary conditions in time, and the other integration variables have twisted (anti-) periodic boundary conditions,

$$\begin{aligned} A(1/T, x) &= A(0, x) - \nabla\chi_n(x) \\ \psi(1/T, x) &= -e^{ie\chi_n(x)} \psi(0, x) \\ \psi^\dagger(1/T, x) &= \psi^\dagger(0, x) e^{-ie\chi_n(x)} \end{aligned} \quad (68)$$

The Gaussian integral over the canonical momentum $E(x)$ is performed to yield, up to a temperature and volume dependent but χ_n -independent overall factor,

$$e^{-S_{\text{eff}}[\chi_n]} = \int \prod_{x \in [0, L]} \prod_{\tau \in [0, 1/T]} d\psi(\tau, x) d\psi^\dagger(\tau, x) dA(\tau, x) e^{-S_E[\psi, \psi^\dagger, A]} \quad (69)$$

where now the Euclidean action is given by

$$S_E = \int_0^{1/T} d\tau \int_0^L dx \left(\frac{1}{2}(\dot{A})^2 - \psi^\dagger [\nabla_\tau + i\alpha\nabla_x + e\alpha A] \psi \right) \quad (70)$$

The boundary conditions can be untwisted by a non-periodic gauge transformation. This is what normally restores A_o , the temporal component of the gauge field to the Euclidean path integral. A suitable non-periodic gauge transformation redefines the integration variables as

$$\begin{aligned} A(\tau, x) &\mapsto A(\tau, x) - \nabla\chi_n(x)T\tau \\ \psi(\tau, x) &\mapsto e^{-ie\chi_n(x)T\tau} \psi(\tau, x) \\ \psi^\dagger(\tau, x) &\mapsto \psi^\dagger(\tau, x) e^{ie\chi_n(x)T\tau} \end{aligned} \quad (71)$$

Note that the spatial boundary conditions for the fermi fields are now changed. The resulting path integral now has the action

$$S_E = \int_0^{1/T} d\tau \int_0^L dx \left(\frac{1}{2}(\dot{A})^2 + \frac{1}{2}T^2(\nabla\chi_n)^2 - \psi^\dagger [\nabla_\tau - ieT\chi_n + i\alpha\nabla_x + e\alpha A] \psi \right) \quad (72)$$

and the boundary conditions are

$$\begin{aligned} \psi(1/T, x) &= -\psi(0, x) & \psi(\tau, L) &= e^{2\pi inT\tau} \psi(\tau, 0) \\ \psi^\dagger(1/T, x) &= -\psi^\dagger(0, x) & \psi^\dagger(\tau, L) &= e^{-2\pi inT\tau} \psi^\dagger(\tau, 0) \\ A(1/T, x) &= A(0, x) & A(\tau, L) &= A(\tau, 0) \\ & & \chi_n(L) &= \chi_n(0) + 2\pi n/e \end{aligned} \quad (73)$$

In order to compute the effective action for $\chi_n(x)$ we must compute the path integral (66) with the Euclidean action (72) and the boundary conditions (73).

The effective electromagnetic field tensor is given by

$$F_{01}(\tau, x) = \dot{A}(\tau, x) - \nabla A_o(\tau, x) \quad (74)$$

where we identify the temporal component of the gauge field in a static gauge ($\nabla_\tau A_o = 0$) as

$$A_o(x) \equiv T\chi_n(x) \quad (75)$$

This field configuration has instanton number n , as it can be seen from the integral

$$-\frac{e}{2\pi} \int_0^{1/T} d\tau \int_0^L dx F_{01}(\tau, x) = n \quad (76)$$

where we have made use of the fact that the field $A(\tau, x)$ has periodic boundary conditions in both τ and x . Thus, the effective vector potential fields in the n 'th sector are just those which are n instantons in a static gauge.

4.1 Z symmetry

We consider the following change of integration variable in the path integral (69, 72):

$$\begin{aligned} \psi(\tau, x) &\mapsto e^{2\pi i k T \tau} \psi(\tau, x) \\ \psi^\dagger(\tau, x) &\mapsto e^{-2\pi i k T \tau} \psi^\dagger(\tau, x) . \end{aligned} \quad (77)$$

When k is an integer, the boundary conditions (73) are unchanged by this substitution and the Jacobian in the path integral measure is one. The net effect is to replace the variable $\chi_n(x)$ by $\chi_n(x) + 2\pi k/e$. Thus, the effective action for χ_n has the symmetry²

$$S_{\text{eff}}[\chi_n] = S_{\text{eff}}[\chi_n + 2\pi k/e] \quad (78)$$

This is a large gauge symmetry analogous to (53) which is associated with the periodic nature of the space in the temporal rather than spatial direction. However, being associated with Euclidean time, it cannot be a basic symmetry of the theory, it is rather an effective symmetry of the Euclidean path integral. We shall presently discuss its interpretation in the Hamiltonian formalism.

In order to obtain a physical interpretation of this symmetry, we consider a modification of electrodynamics where there is an array of static external charges e_1, \dots, e_k located at positions x_1, \dots, x_k . This can be taken into account by a modification of the Gauss' law to

$$-\nabla E(x) - e\psi^\dagger(x)\psi(x) - \sum_{j=1}^k e_j \delta(x - x_j) \sim 0 \quad (79)$$

²Note that this could in principle be only a formal symmetry of the path integral. Here, it survives path integration because of anomaly cancellation, similar to the cancellation of gauge anomalies.

The sole effect of this modification in the partition function (64) is the replacement

$$e^{-S_{\text{eff}}[\chi_n]} \rightarrow e^{-S_{\text{eff}}[\chi_n]} \prod_{j=1}^k e^{-ie_j \chi_n(x_j)} \quad (80)$$

This implies that the k-point correlator

$$\left\langle \prod_{j=1}^k e^{-ie_j \chi(x_j)} \right\rangle \equiv \frac{\sum_n e^{-in\theta} \int [d\chi_n] e^{-S_{\text{eff}}[\chi_n]} \prod_{j=1}^k e^{-ie_j \chi_n(x_j)}}{\sum_n e^{-in\theta} \int [d\chi_n] e^{-S_{\text{eff}}[\chi_n]}} \quad (81)$$

is the ratio of the partition function for the electrodynamic system in the presence of the external charges to the partition function of the same system in the absence of external charges. Thus, the free energy of the system with charges, compared to that without is given by

$$F(e_j, x_j) - F_o = -T \ln \left\langle \prod_{j=1}^k e^{-ie_j \chi(x_j)} \right\rangle \quad (82)$$

where the bracket $\langle \rangle$ is the average over the fields χ_n with the measure $\sum_n \exp(-S_{\text{eff}}[\chi_n] + in\theta)$. Thus, the correlators of the exponential operators measure the Coulomb energy of external charges. In this way they probe the ability of the system to screen external charges.

The symmetry of the effective field theory under the Z transformation, if it is not spontaneously broken, poses a restriction on the correlators which can be non-zero – and therefore it restricts which arrays of external charges can have finite free energy (82). In finite volume, this symmetry is certainly realized canonically and the result is that any expectation value of the form (81) averages to zero when the charges do not add to multiples of the electron charge,

$$\sum_i e_i = me . \quad (83)$$

Whether this symmetry persists in the infinite volume limit $L \rightarrow \infty$, is an interesting question which we shall discuss in following sections.

From the definition (55) of the generators of large gauge transformations, we see that the Z transformation (78) changes the generators by

$$G_l[\chi_n] \mapsto G_l[\chi_n] - 2\pi k \int_0^L dx \psi^\dagger \psi . \quad (84)$$

Accordingly, going back to the definition of the partition function in (62), we see that the Z transformation corresponds to the replacement of the density matrix $e^{-H/T}$ by the operator

$$e^{-H/T} \exp \left(-2\pi i k \int_0^L dx \psi^\dagger \psi \right) \quad (85)$$

in the trace over the physical states. Since all of the physical states obey Gauss' law, with finite L and periodic boundary conditions, they have zero fermion number. Therefore, the exponential of the fermion number is trivially the unit matrix on the space of physical states. The question arises whether this fact persists in the infinite volume limit, or if there are states with arbitrary fermion number.

4.2 Computing the Fermion Determinant

Now we want to calculate the effective action for χ , the time-component of the gauge field in a static gauge (75). For this we first integrate out the fermions from the path integral (66,72).

When the fermion mass is zero, the Dirac operator has zero modes for any of the field configurations with $n \neq 0$. Thus, all the terms in the sum over n except the term with $n = 0$ vanish.

The Dirac operator has zero modes as a consequence of the Atiyah-Singer index theorem (see e.g. Ref. [37]). An explicit demonstration in the Schwinger model can be found in Sachs and Wipf [15]. The Dirac operator has exactly $|n|$ zero modes in the n -instanton sector.

For expectation values of Polyakov lines, we need to take into account only the zero instanton sector, as there are no fermions in our correlators to soak up the zero modes. The fermion determinant in the zero instanton sector is³

$$\int d\bar{\psi} d\psi e^{\int \bar{\psi} \gamma \cdot (\nabla - ieA) \psi} = \det(\gamma \cdot \nabla - ie\gamma \cdot A) \quad (86)$$

We begin with a Hodge decomposition of the gauge field

$$A_\mu = \nabla_\mu \phi + \epsilon_{\mu\nu} \nabla_\nu \Omega + 2\pi h_\mu / e \quad (87)$$

where h_μ are the harmonic modes,

$$h_\mu = \frac{e}{2\pi} \frac{T}{L} \int_0^{1/T} d\tau \int_0^L dx A_\mu(\tau, x) . \quad (88)$$

As the gauge field lies in the zero instanton sector, the pure gauge (ϕ) and coexact (Ω) part have to be strictly periodic in space, c.f. (73).

Using this decomposition, the fermion determinant can be written as

$$\begin{aligned} \det(\gamma \cdot (\nabla - ieA)) &= \det\left(e^{i\phi + \gamma_5 \Omega} \gamma \cdot (\nabla - 2\pi i h) e^{-i\phi + \gamma_5 \Omega}\right) \\ &= \det(\gamma \cdot (\nabla - 2\pi i h)) \det e^{2\gamma_5 \Omega} \end{aligned} \quad (89)$$

Here, we have assumed that the determinant of $e^{i\phi}$ is the inverse of the determinant of $e^{-i\phi}$. This can be shown to be true using a gauge invariant regularization, e.g. Pauli-Villars regularization. The other factor is

$$\det(e^{2\gamma_5 \Omega}) = \exp(\text{tr} 2\gamma_5 \Omega) \quad (90)$$

This is a standard computation; the coexact part of the gauge field carries the chiral anomaly, which can be integrated using any gauge invariant regularization. Noting that $\Delta\Omega = F_{01}$, we get

$$\begin{aligned} \det(e^{2\gamma_5 \Omega}) &= \exp\left(\frac{e^2}{\pi} \int d^2x \Omega(x) F_{01}(x)\right) \\ &= \exp\left(-\frac{e^2}{2\pi} \int d^2x F_{01} \frac{1}{-\nabla^2} F_{01}\right) \end{aligned} \quad (91)$$

³Note that compared to the Minkowskian γ -matrices used in (45), the γ 's used henceforth obey Clifford relations with an Euclidean metric, thus absorbing the extra i in Action (72).

In the field strength F_{01} there is no contribution from the harmonic modes of A_μ .

The part of the determinant of the Dirac operator containing only the harmonic components of the gauge field has the form

$$\det(\gamma \cdot (\nabla - i2\pi h)) = \prod_{m,n} \left[((2n+1)\pi T + 2\pi h_o)^2 + (2\pi m/L + 2\pi h_1)^2 \right] \quad (92)$$

which is the square modulus of the chiral determinant

$$\det D_+ \equiv \prod_{mn} ((2n+1)\pi T + 2\pi h_o + 2\pi im/L + 2\pi ih_1) \quad (93)$$

The latter has the Z -symmetry invariance $h_0 \rightarrow h_0 + kT$, $k \in Z$, and depends only on the complex coordinate $u = h_0 + ih_1$. One can compute the determinant (93) by means of a regularization that preserves the Z -symmetry but breaks the holomorphic factorization of (92) [38]. Namely, one shall obtain for (93) a Z -invariant result that will depend also on the coordinate \bar{u} . Such a result is the well-known expression of (93) in terms of Jacobi theta and Dedekind eta functions,

$$\begin{aligned} \det D_+ &= \frac{1}{\eta(it)} e^{-\pi t \tilde{h}_o^2 + 2\pi i \tilde{h}_o (1/2 - \tilde{h}_1)} \Theta\left(\frac{1}{2} - \tilde{h}_1 + it\tilde{h}_o, it\right) \\ &\equiv \frac{1}{\eta(it)} \Theta\left[\begin{matrix} \tilde{h}_o \\ \frac{1}{2} - \tilde{h}_1 \end{matrix}\right](0, it) \end{aligned} \quad (94)$$

where the modular parameter is $it = iLT$, and the harmonic modes are rescaled with the corresponding length scales to get dimensionless quantities:

$$\tilde{h}_o = h_o/T ; \quad \tilde{h}_1 = Lh_1 . \quad (95)$$

For the theta functions, we follow the labelling conventions of Mumford [39],

$$\Theta(z, it) = \sum_{n \in \mathbb{Z}} \exp\left(-\pi t n^2 + 2\pi i n z\right) \quad (96)$$

$$\Theta\left[\begin{matrix} a \\ b \end{matrix}\right](z, it) = e^{-\pi t a^2 + 2\pi i a(z+b)} \Theta(z + b + ita, it) . \quad (97)$$

As announced the $\tilde{h}_0^2 = (u + \bar{u})^2/4$ term in (94) breaks the holomorphic factorization of the determinant (92). This is a finite local counterterm that can be added to the effective action in order to obtain a gauge invariant result [38]. As a matter of fact, the gauge symmetry on the harmonic component h_0 of the gauge field is nothing but the Z -symmetry. Alternatively, one can maintain the holomorphic factorization of the determinant (92) and loose the Z -symmetry invariance [40]. Our choice is to keep the Z -symmetry invariance.

The total fermion determinant is obtained by combining the coexact piece (91) with the harmonic piece, which is the modulus square of the chiral determinant (94). Expressing the field strength in terms of the spatial gauge field A and the static temporal field χ , we finally get for the effective action

$$e^{-S_{\text{eff}}[\chi]} = \int [dA] \det(i\gamma \cdot D) e^{-\frac{1}{2} \int dx d\tau \{A^2 + T^2 \nabla \chi^2\}}$$

$$\begin{aligned}
&= \int d\tilde{h}_1 \left| \eta(it)^{-1} \Theta \left[\begin{smallmatrix} \tilde{h}_o \\ \frac{1}{2} - \tilde{h}_1 \end{smallmatrix} \right] (0, it) \right|^2 \times \\
&\quad \times \int [d\hat{A}] \exp \left\{ \frac{1}{2} \int dx d\tau \left\{ \hat{A} \left(1 - \frac{m_s^2}{\nabla^2} \right) \nabla_\tau^2 \hat{A} - T^2 \left((\nabla_x \hat{\chi})^2 + m_s^2 \hat{\chi}^2 \right) \right\} \right\}
\end{aligned} \tag{98}$$

The hat on the fields means that the harmonic part is removed. We have denoted the Schwinger mass by $m_s^2 = e^2/\pi$.

The gauge choice (75) has not fixed completely the gauge, since the harmonic part of the gauge field is unaffected by (75). Consequently the Z -symmetry is still present as a residual gauge invariance.

The \tilde{h}_1 and \hat{A} integrations can now be done, yielding (up to normalization)

$$\begin{aligned}
e^{-S_{\text{eff}}[\chi]} &= \sum_{n \in \mathbb{Z}} e^{-2\pi t(n + \tilde{h}_o)^2} \times e^{-T/2 \int_0^L dx \{ (\nabla \chi)^2 + m_s^2 \chi^2 \}} \\
&= \sum_{n \in \mathbb{Z}} \exp \left\{ -T/2 \int_0^L dx \{ (\nabla \chi)^2 + m_s^2 (\chi + 2\pi n/e)^2 \} \right\}
\end{aligned} \tag{99}$$

In order to get a finite result, one has to gauge fix the residual (spatial) large gauge transformations by restricting the integration over \tilde{h}_1 to the period $[0, 1]$ of the integrand.

Equation (99) provides the form of the effective action which explicitly realizes the symmetry (78).

5 Polyakov loop correlators

Now we are in position to calculate expectation values and correlators of Polyakov loops, and accordingly to decide, whether the Z symmetry (78) is spontaneously broken. As indicated in Section 4.1, inserting Polyakov loops probes the response of the theory to static external charges. We take the external charges to have charge \tilde{e} .

Due to the zero modes of the Dirac operator, only the zero instanton sector contributes to the Polyakov loop correlators (81). The expectation value divides in a global and local part:

$$\begin{aligned}
\langle e^{i\tilde{e}\chi(x)} \rangle &= 1/\mathcal{N} \int [d\chi] e^{-S_{\text{eff}}[\chi]} e^{i\tilde{e}\chi(x)} \\
&= \int d\tilde{h}_o \sum_{n \in \mathbb{Z}} e^{-2\pi t(n + \tilde{h}_o)^2 + 2\pi i \tilde{h}_o \tilde{e}/e} \times \int [d\hat{\chi}] e^{-T/2 \int_0^L dy \{ (\nabla \hat{\chi})^2 + m_s^2 \hat{\chi}^2 \}} e^{i\tilde{e}\hat{\chi}(x)}
\end{aligned} \tag{100}$$

The global part is easily calculated by Poisson resummation:⁴

$$\int d\tilde{h}_o \sum_{n \in \mathbb{Z}} e^{-2\pi t(n + \tilde{h}_o)^2 + 2\pi i \tilde{h}_o \tilde{e}/e} = \sqrt{\frac{1}{2t}} \int d\tilde{h}_o \sum_{\nu \in \mathbb{Z}} e^{-\pi/2t \nu^2 + 2\pi i \tilde{h}_o (\tilde{e}/e + \nu)}$$

⁴ Note that as a gauge fixing of large spatial gauge transformations, the domain of the integration has to be restricted to the period of the integrand, which depends on the value of \tilde{e}/e .

$$= \begin{cases} \sqrt{\frac{1}{2t}} e^{-\pi/2t (\tilde{e}/e)^2}, & \text{if } \tilde{e}/e \in \mathcal{Z} \\ 0, & \text{otherwise} \end{cases} \quad (101)$$

The local part can be expressed as

$$\begin{aligned} \int [d\hat{\chi}] e^{i\tilde{e}\chi(x)} e^{-T/2 \int_0^L dy \{ \hat{\chi}(y)(m_s^2 - \nabla^2) \hat{\chi}(y) \}} &\sim e^{-\frac{\tilde{e}^2}{2T} K(x,x)} \\ &= \exp \left\{ \frac{\tilde{e}^2}{2T L m_s^2} - \frac{\tilde{e}^2}{4T m_s} \coth \frac{1}{2} L m_s \right\} \end{aligned} \quad (102)$$

where we used the harmonic oscillator Green's function on the circle (with the contribution of the zero-modes subtracted),

$$K(x, y) = \frac{1}{L} \sum_{n \neq 0} \frac{e^{2\pi i n(x-y)/L}}{m_s^2 + 4\pi^2 n^2/L^2} = \frac{1}{2m_s} \frac{\cosh \left(\frac{1}{2} L m_s (1 - 2|x-y|/L) \right)}{\sinh \frac{1}{2} L m_s} - \frac{1}{L m_s^2}. \quad (103)$$

Combining (101) and (102), we get for the Polyakov loop expectation value

$$\langle e^{i\tilde{e}\chi(x)} \rangle = \begin{cases} \exp \left\{ -\frac{\tilde{e}^2}{4T m_s} \coth \frac{1}{2} L m_s \right\}, & \text{if } \tilde{e}/e \in \mathcal{Z} \\ 0, & \text{otherwise} \end{cases} \quad (104)$$

This is consistent with the expected unbroken realization of the Z symmetry for finite volume. The heat bath screens only external charges that are integer multiples of the electron charge, by bounding dynamical fermions from the heat bath with the external charge. This can be viewed as a proof of the confining nature of finite volume one dimensional electrodynamics.

In infinite volume, one would expect the contribution of the harmonic modes to be trivial. Accordingly, one would expect the nonvanishing value of the Polyakov loop in Equation (104) to be valid irrespective of the value of \tilde{e} , which would be a signal of Z symmetry breaking.

In order properly to investigate the possible symmetry breaking in the infinite volume limit, we need the Polyakov-anti-Polyakov loop correlator. This is again readily calculated using the effective action (99). The harmonic contributions cancel between the Polyakov and anti-Polyakov loops, and we are left with the integration

$$\begin{aligned} \langle e^{i\tilde{e}\chi(x)} e^{-i\tilde{e}\chi(y)} \rangle &= 1/\mathcal{N} \int [d\hat{\chi}] e^{\int_0^L \{ i\tilde{e}\hat{\chi}(x')(\delta(x'-x) - \delta(x'-y)) - T/2 \hat{\chi}(x')(m_s^2 - \nabla^2) \hat{\chi}(x') \}} \\ &= \exp \left\{ -\tilde{e}^2/T (K(x, x) - K(x, y)) \right\} \\ &= \exp \left\{ \frac{-\tilde{e}^2}{2T m_s} \left(\coth \frac{1}{2} L m_s - \frac{\cosh \left(\frac{1}{2} L m_s (1 - 2|x-y|/L) \right)}{\sinh \frac{1}{2} L m_s} \right) \right\} \end{aligned} \quad (105)$$

In the infinite volume limit we get

$$\langle e^{i\tilde{e}\chi(x)} e^{-i\tilde{e}\chi(y)} \rangle \xrightarrow{L \rightarrow \infty} \exp \left\{ -\frac{\tilde{e}^2}{2T m_s} (1 - e^{-m_s|x-y|}) \right\}. \quad (106)$$

This result shows that the system does not cluster decompose:

$$\left\langle e^{i\tilde{e}\chi(x)} e^{-i\tilde{e}\chi(y)} \right\rangle_{L=\infty} \xrightarrow{|x-y|\rightarrow\infty} \exp \left\{ -\frac{\tilde{e}^2}{2Tm_s} \right\} \quad (107)$$

This is consistent with the value of the infinite volume Polyakov loop expectation value anticipated from (104). For $\tilde{e} = e$, our results (104, 106) agree with the results of [24] on Wilson loop expectation values of the zero temperature Schwinger model on a circle, upon a Wick-rotation.

Equation (106) shows that, in the infinite volume limit, there is off diagonal long-range order, and the Z symmetry (78) is thus spontaneously broken. The thermal state of the Schwinger model can screen arbitrary external charges, and it is in a deconfined phase.

We interpret our results as indicating the possible presence of the “bleached” states of Ref. [5] in the spectrum of the Schwinger model. These are states where the charges of the deconfined fermions are completely screened by the thermal state. Our results thus support the screening vs. confinement discussion of Refs. [31, 30, 32].

In Equation (85) we interpreted the Z symmetry as imposing a condition on the allowed fermion numbers of the states in the theory. As the symmetry is broken in infinite volume, we conclude that in infinite volume states with arbitrary fermion number may exist. The screening state, being able to screen *any* charge, not only multiples of the fundamental charge, is a non-local, polarized vacuum like state, and as such it can a priori have any fermion number.

6 Concluding Remarks

In this paper we compute explicitly the effective action and the correlators of the Polyakov loop operator in the one flavor Schwinger model at finite temperature in order to investigate the phases of one dimensional Q.E.D. Our aim is to provide a convincing test of the recent proposal [25] that the Polyakov loop operator is indeed useful to distinguish between a confined and a deconfined phase of an abelian model coupled with fermionic matter.

We present a form of the finite temperature effective action which explicitly realizes the Z symmetry. We show that in one-dimensional Q.E.D. with massless fermions the Z symmetry is not broken in finite volume. The Z -symmetry is broken — due to strong infrared effects — only in the infinite volume limit where there is off-diagonal long range order and the physical states may have arbitrary fermion number.

In infinite volume, the thermal state of the Schwinger model can screen an arbitrary external charge and, therefore, it is in the deconfined phase. Our explicit computation of the Z symmetry breaking in the Schwinger model is supported by two simple arguments providing a sound physical intuition for the breaking of a

discrete symmetry in a one dimensional field theory. The massive Schwinger model, on the other hand, is confining and the Z symmetry is not broken, at least when the temperature is much greater than the electron mass and the confinement scale is set by the dimensional electron charge.

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